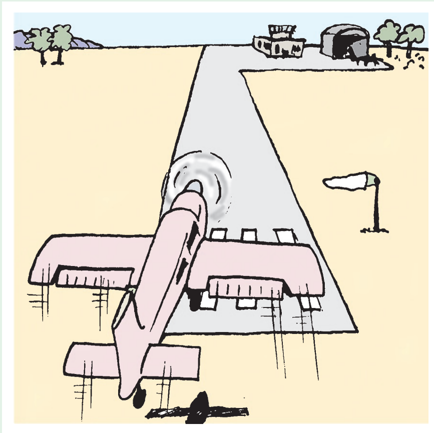
Chapter 26

Vectors

Contents:

OPENING PROBLEM

You have probably seen a film or video of an aeroplane attempting to land with a high cross wind perpendicular to the runway.



Just before the plane touches down, it is held at an angle to the runway. This helps to compensate for the wind, so the overall movement of the plane is parallel to the runway.

Suppose an aeroplane is landing from the west. Its landing speed in still conditions is 150 km/h. The aeroplane experiences a cross wind of 30 km/h from the south.

Things to think about:

a If the plane faced due east, with the wind blowing from the south, what would happen?

b In order to compensate for the wind and land safely:

  i in which direction should the plane face

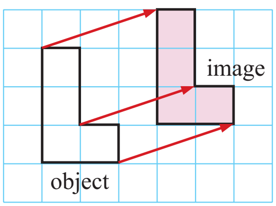
  ii what will the actual speed of the plane be?

Throughout this course we have dealt with many quantities that have size but not direction. For example, we have studied length, area, volume, time, and speed. We call these quantities **scalars**.

Some quantities, such as displacement, acceleration, and momentum, have size *and* direction. We call these quantities **vectors**.

A **vector** is a quantity which has both size and direction.

We have already seen an example of a vector in transformation geometry.



In the translation shown, every point on the object moves under a translation 3 units horizontally and then 1 unit vertically.

By Pythagoras' theorem, each point moves $\sqrt{10}$ units in the particular direction shown in the figure.

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We saw in **Chapter 16** that this translation can be specified by the translation vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

where 3 represents the horizontal movement and
1 represents the vertical movement.

In general, any translation can be specified using a **column vector** $\begin{pmatrix} x \\ y \end{pmatrix}$. The vector describes both the direction of motion and the distance travelled by each point on the object.

- x tells us how far **horizontally** we move and in what direction.
  - $x > 0$ means we move to the *right*.
  - $x < 0$ means we move to the *left*.

- y tells us how far **vertically** we move and in what direction.
  - $y > 0$ means we move *upwards*.
  - $y < 0$ means we move *downwards*.

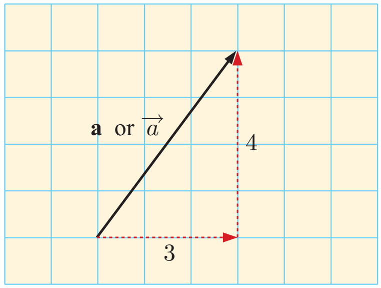
x is often called the **horizontal component** and y the **vertical component** of the translation.

A VECTOR REPRESENTATION

A vector quantity can be represented using a small arrow over a lower case letter.

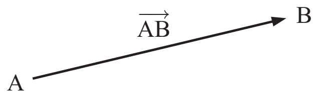
However, in textbooks we use a bold lower case letter.

For example, in the diagram alongside, the illustrated vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is denoted $\mathbf{a}$ or $\overrightarrow{a}$.



We write $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ or $\overrightarrow{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

An arrowhead is used to show the direction of the vector. The non-arrow end is the **start** of the vector, and the arrowhead end is the **end** of the vector.

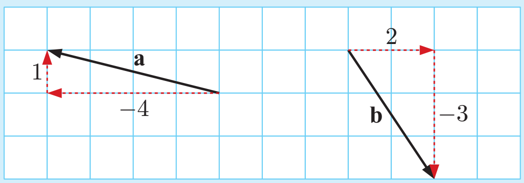


Another way to represent a vector is by referring to its end points.

If we label the end points A and B, then $\overrightarrow{AB}$ is the vector from point A to point B.

Example 1 **Self Tutor**

Represent $\mathbf{a} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ on grid paper.



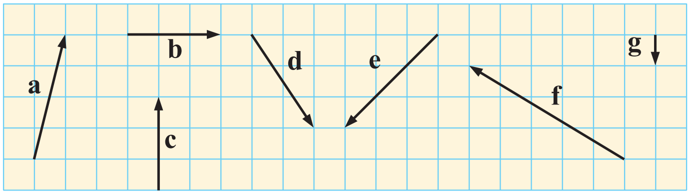
EXERCISE 26A

1 On grid paper, draw these vectors:

a $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

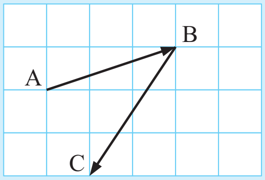
e $\mathbf{e} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **f** $\mathbf{f} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ **g** $\mathbf{g} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$ **h** $\mathbf{h} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

2 Write each vector in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:



Example 2 — Self Tutor

Write $\overrightarrow{AB}$ and $\overrightarrow{BC}$ in component form.



To get from A to B we move 3 units right and 1 unit up.

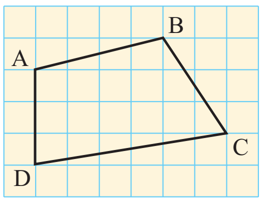
$\therefore \quad \overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

To get from B to C we move 2 units left and 3 units down.

$\therefore \quad \overrightarrow{BC} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

3 For the given figure, write the following as column vectors:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BC}$ **c** $\overrightarrow{CD}$ **d** $\overrightarrow{DA}$
e $\overrightarrow{BA}$ **f** $\overrightarrow{AC}$ **g** $\overrightarrow{BD}$ **h** $\overrightarrow{AA}$



B THE LENGTH OF A VECTOR

The **length** of a vector **a** is the distance between its starting point and its end point. It is written $|\mathbf{a}|$.

The length is also called the **magnitude** of the vector.

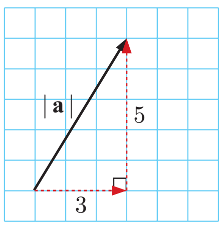
To find the length of a vector we can use **Pythagoras' theorem**.

For example, consider the vector $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Now $|\mathbf{a}|^2 = 3^2 + 5^2$ {Pythagoras}

$\therefore \quad |\mathbf{a}| = \sqrt{3^2 + 5^2}$ {as $|\mathbf{a}| > 0$}

$= \sqrt{34}$ units



The vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ has length $|\,\mathbf{a}\,| = \sqrt{{a_1}^2 + {a_2}^2}$.

Example 3 — Self Tutor

Find the length of: **a** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$

a The length of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$
$= \sqrt{2^2 + 1^2}$
$= \sqrt{4 + 1}$
$= \sqrt{5}$ units

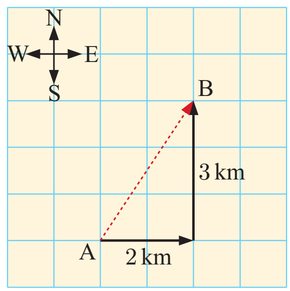
b The length of $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$
$= \sqrt{5^2 + (-4)^2}$
$= \sqrt{25 + 16}$
$= \sqrt{41}$ units

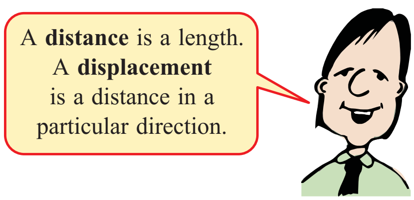
DISTANCE AND DISPLACEMENT

A ship starts at point A. It sails to point B which is 2 km to the east and 3 km to the north of A.

The **displacement vector** of the ship is $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

The **distance** travelled by the ship is $|\,\overrightarrow{AB}\,| = \sqrt{2^2 + 3^2}$
$= \sqrt{13}$ km.





EXERCISE 26B

1 Find the length of:

a $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

e $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ **f** $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ **g** $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ **h** $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$

i $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ **j** $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$ **k** $\begin{pmatrix} 10 \\ -1 \end{pmatrix}$ **l** $\begin{pmatrix} 8 \\ -2 \end{pmatrix}$

2 Find the length of:

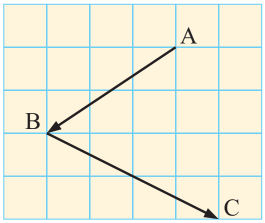
a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ **d** $\begin{pmatrix} -3 \\ -4 \end{pmatrix}$

Comment on your answers.

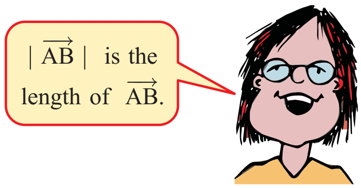
3 Brigitta enjoys hiking. One day on a hike, she finishes 4 km east and 6 km south of her starting point.

- **a** Write Brigitta's displacement vector.
- **b** In a straight line, how far is Brigitta from her starting point?

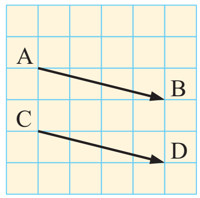
4



- **a** Write $\overrightarrow{AB}$ in component form, and hence find $|\overrightarrow{AB}|$.
- **b** Write $\overrightarrow{BC}$ in component form, and hence find $|\overrightarrow{BC}|$.



5
- **a** What geometrical property do vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have?
- **b** Write in component form:
  - **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{CD}$
- **c** Find the lengths of $\overrightarrow{AB}$ and $\overrightarrow{CD}$.



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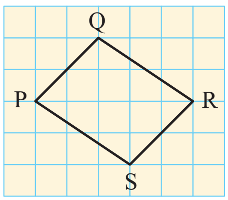


- **a** Find $\overrightarrow{PQ}$ and $\overrightarrow{SR}$, and find their lengths.
- **b** Find $\overrightarrow{RQ}$ and $\overrightarrow{SP}$, and find their lengths.
- **c** Use **a** and **b** to classify quadrilateral PQRS.

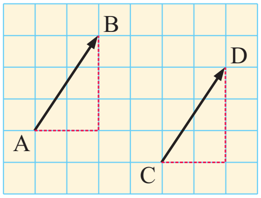
7 $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ are called 'opposite vectors'.

- **a** Illustrate each vector on grid paper.
- **b** Explain why the vectors are called opposite vectors.
- **c** True or false? "Opposite vectors have the same length."

C EQUAL VECTORS

Two vectors are **equal** if they have the same x and y-components.

If $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$, then $a = c$ and $b = d$.



In the given figure, $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal vectors, and we write $\overrightarrow{AB} = \overrightarrow{CD}$.

Notice that equal vectors have the same length and direction. They are parallel.

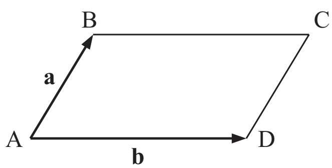
EXERCISE 26C

1 Explain why $\begin{pmatrix} 5 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

2 What can be deduced if $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$?

3 Find a and b such that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^2 \\ -b \end{pmatrix}$.

4



ABCD is a parallelogram with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.
State, with reasons, vector expressions for $\overrightarrow{DC}$ and $\overrightarrow{BC}$.

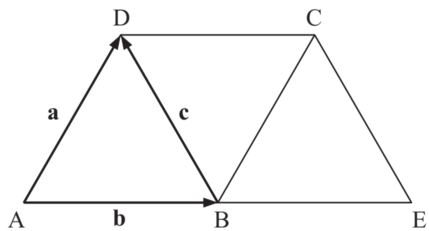
5 Triangles ABD, BEC, and BCD are equilateral.
Suppose $\overrightarrow{AD} = \mathbf{a}$, $\overrightarrow{AB} = \mathbf{b}$, and $\overrightarrow{BD} = \mathbf{c}$.

a Find, in terms of vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, vectors representing:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{BE}$ **iii** $\overrightarrow{DC}$ **iv** $\overrightarrow{EC}$

b Explain why $\mathbf{a} \neq \mathbf{b}$.

c Does $|\mathbf{a}| = |\mathbf{b}|$?

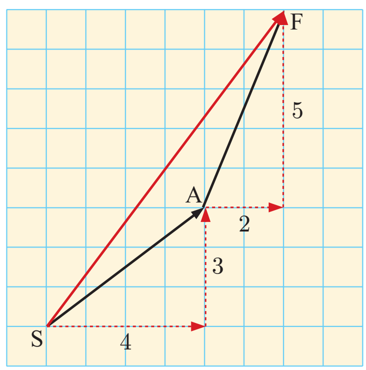


D VECTOR ADDITION

Suppose a walker starts at point S and walks to point A with displacement vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

The walker then walks from A to a final point F, this time with displacement vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

The walker's final displacement from S is given by the displacement vector $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$.



This final displacement can also be found by an **addition of vectors**:

$$\begin{pmatrix}4\\3\end{pmatrix}+\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}4+2\\3+5\end{pmatrix}=\begin{pmatrix}6\\8\end{pmatrix}.$$

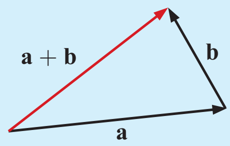
If $\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$ then $\mathbf{a}+\mathbf{b}=\begin{pmatrix}a_1+b_1\\a_2+b_2\end{pmatrix}$.

To find $\mathbf{a}+\mathbf{b}$ geometrically:

Step 1: Draw vector **a**.

Step 2: At the arrow end of **a**, draw vector **b**.

Step 3: Draw a vector from the start of **a** to the end of **b**. The resultant vector is $\mathbf{a}+\mathbf{b}$.



DEMO

Example 4 Self Tutor

If $\mathbf{a}=\begin{pmatrix}2\\5\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}4\\-1\end{pmatrix}$, and $\mathbf{c}=\begin{pmatrix}-2\\-3\end{pmatrix}$, find:

a $\mathbf{a}+\mathbf{b}$ **b** $\mathbf{b}+\mathbf{c}$ **c** $\mathbf{a}+\mathbf{b}+\mathbf{c}$

a $\mathbf{a}+\mathbf{b}$

$$=\begin{pmatrix}2\\5\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}$$
$$=\begin{pmatrix}2+4\\5+-1\end{pmatrix}$$
$$=\begin{pmatrix}6\\4\end{pmatrix}$$

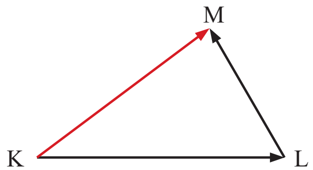
b $\mathbf{b}+\mathbf{c}$

$$=\begin{pmatrix}4\\-1\end{pmatrix}+\begin{pmatrix}-2\\-3\end{pmatrix}$$
$$=\begin{pmatrix}4+-2\\-1+-3\end{pmatrix}$$
$$=\begin{pmatrix}2\\-4\end{pmatrix}$$

c $\mathbf{a}+\mathbf{b}+\mathbf{c}$

$$=\begin{pmatrix}2\\5\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}+\begin{pmatrix}-2\\-3\end{pmatrix}$$
$$=\begin{pmatrix}2+4+-2\\5+-1+-3\end{pmatrix}$$
$$=\begin{pmatrix}4\\1\end{pmatrix}$$

The points K, L, and M form the vertices of a triangle. Suppose we wish to go from K to M.



We could go from K to M directly along vector $\overrightarrow{KM}$.

Alternatively, we could go from K to L, and then from L to M. In this case we add the two vectors $\overrightarrow{KL}$ and $\overrightarrow{LM}$, giving $\overrightarrow{KL}+\overrightarrow{LM}$.

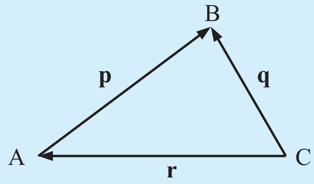
The result is the same either way, so $\overrightarrow{KL}+\overrightarrow{LM}=\overrightarrow{KM}$.

same letter

We can use the same method to add any number of vectors.

Example 5 — Self Tutor

Write a vector equation to connect vectors **p**, **q**, and **r**.



We can move from C to B directly along vector **q**.

Alternatively, we can go from C to A, and then from A to B. In this case we have $\mathbf{r}+\mathbf{p}$.

$\therefore \quad \mathbf{q}=\mathbf{r}+\mathbf{p}$.

Example 6 — Self Tutor

Simplify:

a $\overrightarrow{LM}+\overrightarrow{MN}$ **b** $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}$

a $\overrightarrow{LM}+\overrightarrow{MN}=\overrightarrow{LN}$ **b** $\overrightarrow{AB}+\overrightarrow{BD}+\overrightarrow{DC}=\overrightarrow{AC}$

EXERCISE 26D

1 Find:

a $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}4\\7\end{pmatrix}$ **b** $\begin{pmatrix}5\\2\end{pmatrix}+\begin{pmatrix}6\\9\end{pmatrix}$ **c** $\begin{pmatrix}3\\-2\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}$

d $\begin{pmatrix}5\\0\end{pmatrix}+\begin{pmatrix}-1\\-3\end{pmatrix}$ **e** $\begin{pmatrix}-7\\2\end{pmatrix}+\begin{pmatrix}4\\-5\end{pmatrix}$ **f** $\begin{pmatrix}-6\\1\end{pmatrix}+\begin{pmatrix}-2\\-3\end{pmatrix}$

g $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}2\\4\end{pmatrix}$ **h** $\begin{pmatrix}5\\-1\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}-2\\-4\end{pmatrix}$

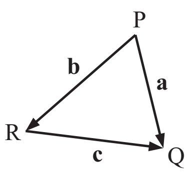
2 If $\mathbf{a}=\begin{pmatrix}5\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\-3\end{pmatrix}$, and $\mathbf{c}=\begin{pmatrix}-4\\1\end{pmatrix}$, find:

a $\mathbf{a}+\mathbf{b}$ **b** $\mathbf{b}+\mathbf{a}$ **c** $\mathbf{b}+\mathbf{c}$ **d** $\mathbf{c}+\mathbf{a}$

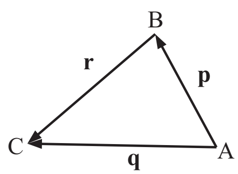
e $\mathbf{a}+\mathbf{a}$ **f** $\mathbf{b}+\mathbf{b}$ **g** $\mathbf{c}+\mathbf{c}+\mathbf{c}$ **h** $\mathbf{a}+\mathbf{b}+\mathbf{c}$

3 Write a vector equation to connect the vectors in:

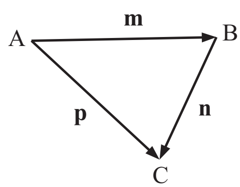
a



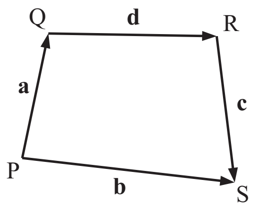
b



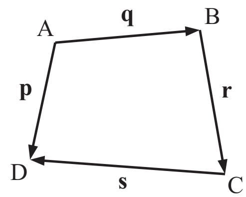
c



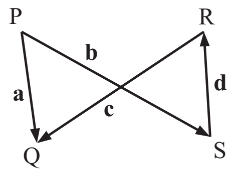
d



e



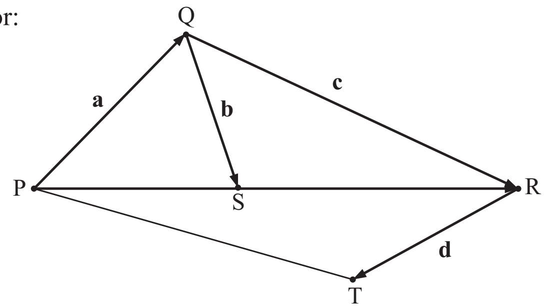
f



4 Write an expression in terms of **a**, **b**, **c**, and **d**, for:

a $\overrightarrow{PS}$ **b** $\overrightarrow{PR}$

c $\overrightarrow{QT}$ **d** $\overrightarrow{PT}$



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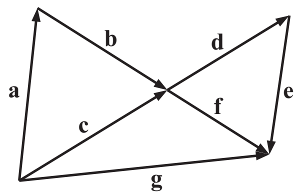


Simplify:

a **a** + **b** **b** **d** + **e**

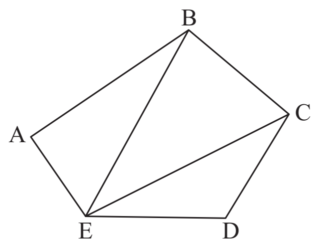
c **c** + **f** **d** **a** + **b** + **f**

e **c** + **d** + **e**

6 Simplify:

a $\overrightarrow{AB} + \overrightarrow{BE}$ **b** $\overrightarrow{BC} + \overrightarrow{CE}$

c $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$ **d** $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$



7 Simplify:

a $\overrightarrow{AP} + \overrightarrow{PB}$ **b** $\overrightarrow{PX} + \overrightarrow{XY} + \overrightarrow{YQ}$ **c** $\overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{ND}$

d $\overrightarrow{SP} + \overrightarrow{PQ} + \overrightarrow{QN}$ **e** $\overrightarrow{EF} + \overrightarrow{FD} + \overrightarrow{DQ} + \overrightarrow{QR}$ **f** $\overrightarrow{CX} + \overrightarrow{XN} + \overrightarrow{ND} + \overrightarrow{DP}$

8 Alongside is a hole at Hackers Golf Club.

a Jack tees off from T and his ball finishes at A. Write a vector to describe the displacement of the ball from T to A.

b Jack plays his second stroke from A to B. Write a vector to describe the displacement of the ball from this shot.

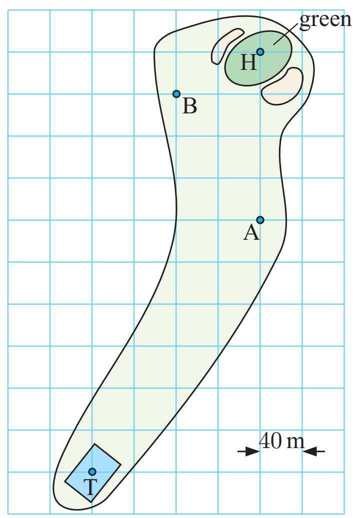
c By great luck, Jack's next shot finishes in the hole H. Write a vector which describes this shot.

d Find:

i $\overrightarrow{TA} + \overrightarrow{AB} + \overrightarrow{BH}$ **ii** $\overrightarrow{TH}$

Comment on your answers.

e Find the straight line distance between the tee T and the hole H.



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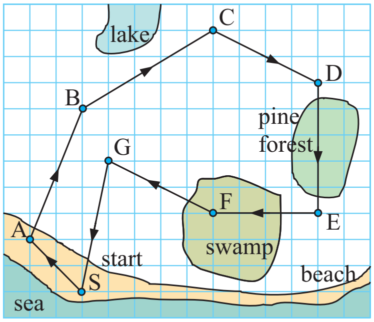


The diagram alongside shows an orienteering course run by Kahu.

a Write a column vector to describe each leg of the course.

b Find the sum of all of the vectors.

c What does the sum in **b** tell us?

ACTIVITY — A VECTOR CIRCUIT

What to do:

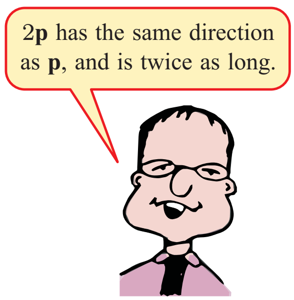
1 Design an orienteering course like that in question **9** of the **Exercise** above. It must contain at least *six* legs and may cross over itself. It must finish where it started.

2 Write each leg in vector form.

3 Explain why the sum of all of the vectors will always be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

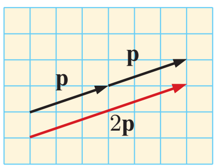
E MULTIPLYING VECTORS BY A NUMBER

If $\mathbf{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ then $\mathbf{p} + \mathbf{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.



If we define $\mathbf{p} + \mathbf{p}$ to be $2\mathbf{p}$, we notice that

$$2\mathbf{p} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 \\ 2 \times 1 \end{pmatrix}.$$



Examples like this suggest that: If $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ then $k\mathbf{p} = \begin{pmatrix} ka \\ kb \end{pmatrix}$.

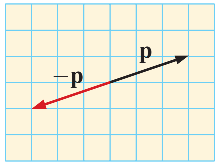
Multiplying a vector by a number in this way is called **scalar multiplication**. This is because the constant k is a scalar.

NEGATIVE VECTORS

If we multiply the vector $\mathbf{p} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ by -1,

we get $(-1)\mathbf{p} = \begin{pmatrix} -1 \times 3 \\ -1 \times 1 \end{pmatrix}$

$\therefore \quad -\mathbf{p} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$



This is the **negative vector** of **p**.

$-\mathbf{p}$ has the same length as **p**, but is in the opposite direction to **p**.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$ is the **negative vector** of **a**.

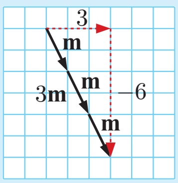
Example 7 **Self Tutor**

If $\mathbf{m} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, find:

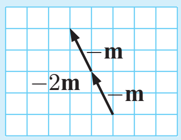
a $3\mathbf{m}$ **b** $-2\mathbf{m}$

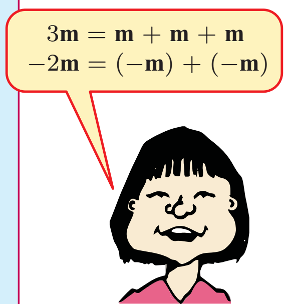
Illustrate your answers.

a $3\mathbf{m} = 3\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
$= \begin{pmatrix} 3 \times 1 \\ 3 \times -2 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ -6 \end{pmatrix}$



b $-2\mathbf{m} = -2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
$= \begin{pmatrix} -2 \times 1 \\ -2 \times -2 \end{pmatrix}$
$= \begin{pmatrix} -2 \\ 4 \end{pmatrix}$





THE ZERO VECTOR

The **zero vector** is the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

It can be obtained by multiplying any vector by the scalar zero.

For example, $0\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \times 2 \\ 0 \times 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The **zero vector**, **0** has length 0.

It is the only vector with no direction.

For any vector **a**, $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.

EXERCISE 26E

1 **a** If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, find $\mathbf{a} + \mathbf{a} + \mathbf{a}$.

b Show using the rule above that $3\mathbf{a}$ gives the same result.

2 For $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, and $\mathbf{d} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, find:

a $2\mathbf{a}$ **b** $4\mathbf{a}$ **c** $-\mathbf{a}$ **d** $3\mathbf{b}$ **e** $-2\mathbf{b}$
f $\frac{1}{2}\mathbf{c}$ **g** $\frac{1}{3}\mathbf{d}$ **h** $-\frac{3}{2}\mathbf{c}$ **i** $-6\mathbf{d}$ **j** $-\frac{1}{2}\mathbf{d}$
k $3\mathbf{a} + \mathbf{b}$ **l** $2\mathbf{b} + \mathbf{d}$ **m** $\mathbf{c} + 4\mathbf{a}$ **n** $2\mathbf{b} + 3\mathbf{c}$ **o** $5\mathbf{d} + 4\mathbf{b}$

3 For $\mathbf{m} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, draw on grid paper:

a $\mathbf{m}$ **b** $2\mathbf{m}$ **c** $-2\mathbf{m}$ **d** $-3\mathbf{m}$ **e** $\frac{1}{2}\mathbf{m}$ **f** $-\frac{1}{3}\mathbf{m}$

4 If $\mathbf{a} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$, find:

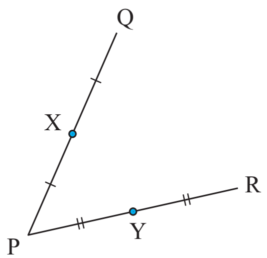
a $3\mathbf{a}$ **b** $-\mathbf{a}$ **c** $\mathbf{a} + \mathbf{0}$ **d** $\mathbf{0} + \mathbf{a}$ **e** $-\mathbf{a} + \mathbf{a}$ **f** $\mathbf{a} + (-\mathbf{a})$

5 Show that for any vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$:

a $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$ **b** $\mathbf{a} + (-\mathbf{a}) = -\mathbf{a} + \mathbf{a} = \mathbf{0}$

6 If $\overrightarrow{AB} = \mathbf{p}$, write $\overrightarrow{BA}$ in terms of $\mathbf{p}$.

7



In the diagram alongside, X is the midpoint of [PQ], and Y is the midpoint of [PR].

Let $\overrightarrow{PX} = \mathbf{a}$ and $\overrightarrow{PY} = \mathbf{b}$.

Find, in terms of **a** and **b**:

a $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$ **c** $\overrightarrow{XP}$
d $\overrightarrow{YP}$ **e** $\overrightarrow{QP}$ **f** $\overrightarrow{RP}$

DISCUSSION — MULTIPLYING VECTORS BY NUMBERS

Jason said that "multiplying a vector by a number does not change its direction, it just makes the vector longer or shorter."

Discuss the inaccuracy of Jason's statement by considering multiplication by 2, $\frac{1}{2}$, -2, and $-\frac{1}{2}$. Correct Jason's statement.

F VECTOR SUBTRACTION

To subtract a vector we add its negative.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= \mathbf{a} + (-\mathbf{b}) \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}\end{aligned}$$

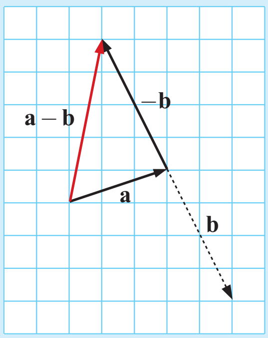
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

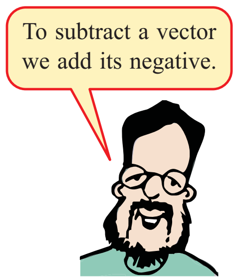
Example 8 **Self Tutor**

If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ find $\mathbf{a} - \mathbf{b}$.

Illustrate how to find $\mathbf{a} - \mathbf{b}$ geometrically.

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 3 - 2 \\ 1 - -4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 5 \end{pmatrix}\end{aligned}$$





EXERCISE 26F

1 Find:

a $\begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 6 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

d $\begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **f** $\begin{pmatrix} 5 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 \\ 0 \end{pmatrix}$

g $\begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **h** $\begin{pmatrix} 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ **i** $\begin{pmatrix} -5 \\ -8 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

2 For the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -5 \\ 0 \end{pmatrix}$, find:

a $\mathbf{a} - \mathbf{b}$ **b** $\mathbf{b} - \mathbf{c}$ **c** $\mathbf{c} - \mathbf{a}$ **d** $2\mathbf{a} - \mathbf{c}$
e $\mathbf{a} + \mathbf{b} - \mathbf{c}$ **f** $\mathbf{b} + 2\mathbf{c} - \mathbf{a}$ **g** $\mathbf{a} - \frac{1}{2}\mathbf{b}$ **h** $2\mathbf{a} + \mathbf{b} - 3\mathbf{c}$

Example 9 Self Tutor

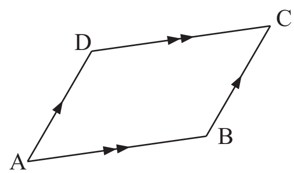
Simplify $\overrightarrow{PQ} - \overrightarrow{RQ}$.

$\overrightarrow{PQ} - \overrightarrow{RQ}$
$= \overrightarrow{PQ} + -\overrightarrow{RQ}$
$= \overrightarrow{PQ} + \overrightarrow{QR}$ $\{\overrightarrow{QR} = -\overrightarrow{RQ}\}$
$= \overrightarrow{PR}$

3 Simplify:

a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{QP} - \overrightarrow{RP}$
c $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$ **d** $\overrightarrow{PQ} - \overrightarrow{RQ} + \overrightarrow{RS} - \overrightarrow{TS} + \overrightarrow{TV}$

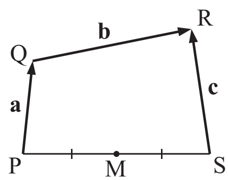
4 ABCD is a parallelogram in which $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.



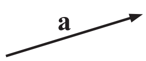
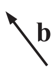
Find:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{BD}$

5 M is the midpoint of [PS]. $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$, and $\overrightarrow{SR} = \mathbf{c}$.
Find, in terms of **a**, **b**, and **c**:

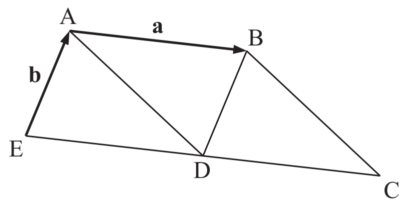


a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$ **d** $\overrightarrow{PM}$

6 For  and  , draw vector diagrams for:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $\mathbf{a} - 2\mathbf{b}$

7 ABDE and ABCD are parallelograms. Find, in terms of **a** and **b**, vector expressions for:

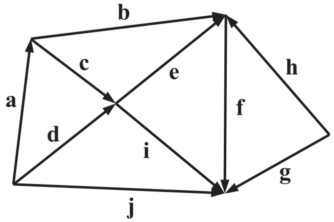


a $\overrightarrow{ED}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{DB}$
d $\overrightarrow{AD}$ **e** $\overrightarrow{BC}$ **f** $\overrightarrow{EC}$

8 Use the diagram to simplify:

a $\mathbf{a}+\mathbf{c}$ **b** $\mathbf{h}+\mathbf{f}$ **c** $\mathbf{j}-\mathbf{i}$

d $\mathbf{d}-\mathbf{c}$ **e** $\mathbf{e}-\mathbf{b}$ **f** $-\mathbf{f}-\mathbf{h}$



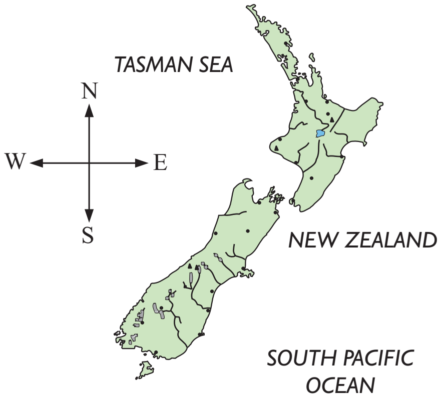
G THE DIRECTION OF A VECTOR

In displacement problems we often need to find the direction of a vector.

One way to measure the direction of travel is to use **true bearings**.

We saw in **Chapter 13** that a true bearing is a measurement of the **clockwise** angle from the **true north direction**.

Remember that the bearing of A from B and the bearing of B from A always differ by 180°.

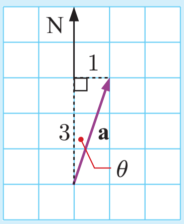


Example 10 **Self Tutor**

Find the angle that the given vector makes with true north:

a $\mathbf{a}=\begin{pmatrix}1\\3\end{pmatrix}$ **b** $\mathbf{b}=\begin{pmatrix}3\\-4\end{pmatrix}$

a



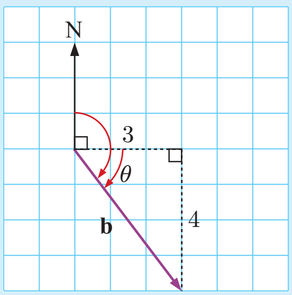
$\tan\theta=\frac{1}{3}$

$\therefore\ \theta=\tan^{-1}\left(\frac{1}{3}\right)$

$\approx 18.4°$

$\therefore$ **a** has the bearing 018.4°.

b



$\tan\theta=\frac{4}{3}$

$\therefore\ \theta=\tan^{-1}\left(\frac{4}{3}\right)$

$\approx 53.1°$

and $\theta+90°\approx 143.1°$

$\therefore$ **b** has the bearing 143.1°.

SPEED AND VELOCITY

We have seen previously how **speed** describes how fast something is moving.

Velocity is a vector which describes speed in a given direction.

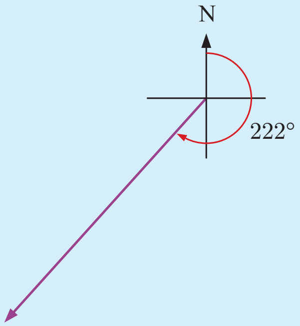
Example 11 **Self Tutor**

Henri is cycling at a constant speed of 40 km/h in the direction 222°.

a Draw an accurate scale diagram showing this information.

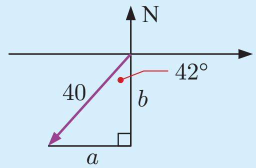
b Find Henri's velocity vector.

a



Scale: 1 mm ≡ 1 km/h

b

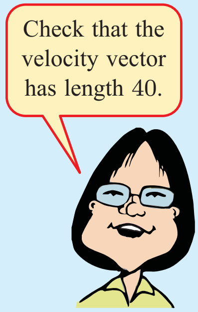


$\sin 42° = \dfrac{a}{40} \quad \therefore \; a = 40 \sin 42°$

$\cos 42° = \dfrac{b}{40} \quad \therefore \; b = 40 \cos 42°$

So, the velocity vector is

$$\begin{pmatrix} -40\sin 42° \\ -40\cos 42° \end{pmatrix} \approx \begin{pmatrix} -26.8 \\ -29.7 \end{pmatrix}$$



EXERCISE 26G

1 Using a diagram only, find the bearing of:

a $\mathbf{a} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

2 Using a diagram only, find the bearing of:

a $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$

3 Use right angled triangle trigonometry to find the bearing of:

a $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

4 Find the length and direction of:

a $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **b** $\mathbf{b} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **c** $\mathbf{c} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **d** $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

5 Jasmin walks at a constant speed of 5 km/h in the direction 057°.

a Draw an accurate scale diagram showing this information.

b Find Jasmin's velocity vector.

6 Chai jogs at a constant speed of 12 km/h in the direction 146°.

a Draw an accurate scale diagram showing this information.

b Find Chai's velocity vector.

7 Jiji walks for 8 km in the direction 303°.

a Draw an accurate scale diagram showing this information.

b Find Jiji's displacement vector.

H PROBLEM SOLVING BY VECTOR ADDITION

In this Section we use vector addition to help solve problems where there are different vector components.

Example 12 Self Tutor

In still conditions, a bird flies at 10 m/s. One day it faces east and tries to fly at its usual speed. What will its actual velocity be if it encounters a wind of 1 m/s from:

a the west **b** the east **c** the north?

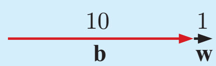
The bird's actual velocity will be the vector addition of its velocity in still conditions, plus the wind.

If there was no wind, the bird's velocity would be $\mathbf{b} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$.

If the wind has velocity vector $\mathbf{w}$, then the bird's actual velocity is $\mathbf{v} = \mathbf{b} + \mathbf{w}$.

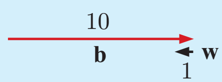
a $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ 0 \end{pmatrix}$

The bird's velocity is 11 m/s to the east.



b $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$

The bird's velocity is 9 m/s to the east.

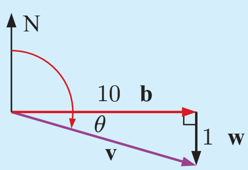


c $\mathbf{w} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so $\mathbf{v} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \end{pmatrix}$

$|\mathbf{v}| = \sqrt{10^2 + (-1)^2}$
$= \sqrt{101}$
≈ 10.05

$\tan\theta = \frac{1}{10}$
$\therefore \theta = \tan^{-1}\left(\frac{1}{10}\right)$
$\approx 5.7°$

So, the bird's velocity is 10.05 m/s on the bearing 095.7°.



not to scale

EXERCISE 26H

1 Quang runs 10 km to the north, and then 5 km to the west.

a Write each part of this run in component vector form.

b Find Quang's displacement vector from his starting point.

c Find Quang's distance from his starting point.

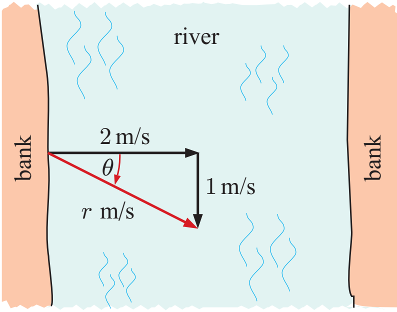
d Find Quang's bearing from his starting point.

2 Aleksandra drives with displacement vector $\begin{pmatrix}14\\2\end{pmatrix}$. She then changes direction and drives with displacement vector $\begin{pmatrix}3\\-11\end{pmatrix}$. Units are in kilometres.

- **a** Illustrate Aleksandra's movement on grid paper.
- **b** Find Aleksandra's displacement vector from her starting point.
- **c** How far is Aleksandra from her starting point?
- **d** What is Aleksandra's bearing from her starting point?

3 The diagram shows a river running from north to south at 1 m/s. A swimmer attempts to swim directly out from the bank at 2 m/s.

- **a** What will the actual speed of the swimmer be?
- **b** Find θ.
- **c** On what bearing will the swimmer actually be heading?



4 The Interislander ferry is steaming due east across Cook Strait at a speed of 20 km/h. Johanna is a passenger on the ferry. She walks from the front of the ferry towards the back at a speed of 5 km/h.

- **a** Find Johanna's resultant speed.
- **b** Is it possible for Johanna to move faster than the ferry? Explain your answer.

5 A yacht is sailing at 14 km/h on the bearing 045°. It is suddenly hit by the wake of a large ship, which pushes it at 4 km/h on the bearing 135°.

- **a** Draw scale diagrams to show the:
  - **i** yacht's original velocity vector
  - **ii** wake's velocity vector.
- **b** Show that:
  - **i** the yacht's original velocity vector was $\begin{pmatrix}7\sqrt{2}\\7\sqrt{2}\end{pmatrix}$
  - **ii** the wake's velocity vector is $\begin{pmatrix}2\sqrt{2}\\-2\sqrt{2}\end{pmatrix}$.
- **c** Assuming that the yacht's captain makes no allowance for the wake, find the new speed and direction of the yacht.

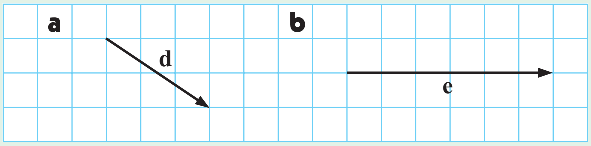
6 An aeroplane takes off from Changi airport in Singapore. Its flight is affected by a 50 km/h wind on the bearing 305°. The aeroplane's *actual* velocity is 350 km/h on the bearing 035°.

- **a** Draw a vector diagram to represent the situation.
- **b** Calculate the velocity vector that the aeroplane would have if it was not affected by the wind.

REVIEW SET 26A

1 On grid paper draw the vectors $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

2 Write in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:



3 Suppose $\mathbf{a} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

a Draw a diagram which shows how to find $\mathbf{a} + \mathbf{b}$.

b Find:

i $\mathbf{c} + \mathbf{b}$ **ii** $\mathbf{a} - \mathbf{b}$ **iii** $\mathbf{a} + \mathbf{b} - \mathbf{c}$

4 Find the length of:

a $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ **c** $\begin{pmatrix} -8 \\ -3 \end{pmatrix}$

5 Given that $\begin{pmatrix} 2a+1 \\ 5 \end{pmatrix} = \begin{pmatrix} b \\ b-2 \end{pmatrix}$, find a and b.

6 Consider $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -3 \\ -5 \end{pmatrix}$.

a Sketch $3\mathbf{p}$.

b Calculate:

i $2\mathbf{q}$ **ii** $3\mathbf{p} + 2\mathbf{q}$ **iii** $\mathbf{p} - 2\mathbf{q}$

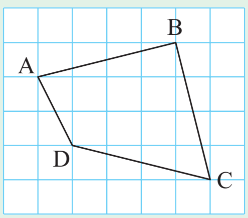
c Draw a diagram which shows how to find $\mathbf{q} + 2\mathbf{p}$.

7 Consider the vector $\mathbf{m} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

a Illustrate the vector on grid paper. **b** Find $|\mathbf{m}|$.

c Find the bearing of the vector.

8



a Find in component form:

i $\overrightarrow{BC}$ **ii** $\overrightarrow{BD}$

b Simplify $\overrightarrow{AD} + \overrightarrow{DC}$.

c Find $|\overrightarrow{AC}|$.

9 Find the bearing of:

a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$

10 A plane is flown 200 km on the bearing 045°, and then 500 km on the bearing 135°.

a Draw a vector diagram of the plane's flight.

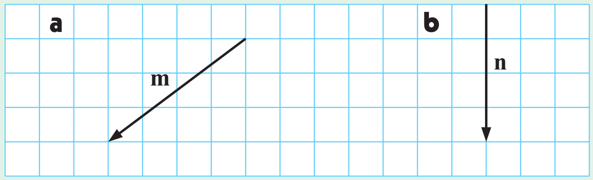
b Calculate how far the plane is now from its starting point.

c On what bearing would the plane need to fly to return directly to its starting point?

REVIEW SET 26B

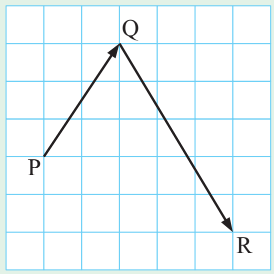
1 On grid paper, draw the vectors $\mathbf{p} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

2 Write in the form $\begin{pmatrix} x \\ y \end{pmatrix}$:



3 **a** Write $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ in component form.

b Find $|\overrightarrow{PQ}|$ and $|\overrightarrow{QR}|$.



4 Find:

a $\begin{pmatrix} 5 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -1 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} 8 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

5 Suppose $\mathbf{a} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$. Find:

a $3\mathbf{a}$ **b** $-2\mathbf{a}$ **c** $-\mathbf{a}$ **d** $0\mathbf{a}$

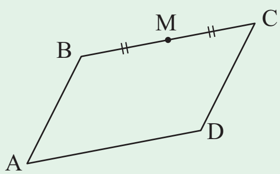
6 Suppose $\mathbf{d} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

a Draw a vector diagram to illustrate $\mathbf{d} - \mathbf{e}$.

b Find $\mathbf{d} - \mathbf{e}$ in component form.

c Find: **i** $2\mathbf{e} + 3\mathbf{d}$ **ii** $4\mathbf{d} - 3\mathbf{e}$

7



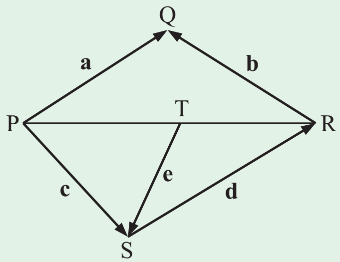
$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{BC} = \mathbf{q}$, and ABCD is a parallelogram. Find a vector expression for:

a $\overrightarrow{CD}$ **b** $\overrightarrow{BM}$ **c** $\overrightarrow{MD}$ **d** $\overrightarrow{AD}$

8 Find in terms of **a**, **b**, **c**, **d**, and **e**:

a $\overrightarrow{TR}$ **b** $\overrightarrow{PR}$

c $\overrightarrow{PT}$ **d** $\overrightarrow{TQ}$



9 Ingrid walks at 5 km/h in the direction 315°. Find Ingrid's velocity vector.

10 A man starts rowing his boat due east towards an island 850 m away. Since he is facing the wrong way, he does not realise a current from the north is pushing him off course. The man can row at 1.5 m/s in still water, and the current pushes him at 0.3 m/s.

a Draw a vector diagram illustrating the man's actual velocity.

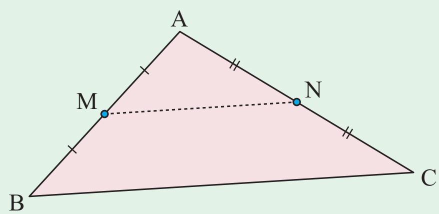
b Find the man's actual speed and bearing.

c If there was no current, how long would it take for the man to reach the island?

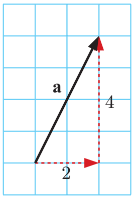
d By how far will the man miss the point on the island he was trying to land at?

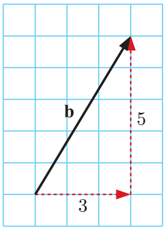
11 Suppose M is the midpoint of [AB] and N is the midpoint of [AC]. The **midpoint theorem** states that [MN] is parallel to [BC] and half its length.

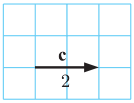
Use vector methods to prove the midpoint theorem.

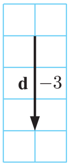


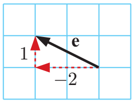
EXERCISE 26A

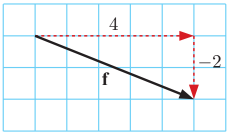
1 **a**


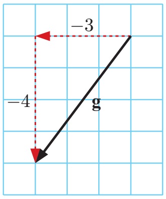
b


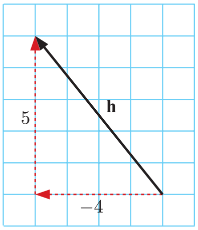
c


d


e


f


g


h


2 $\mathbf{a}=\begin{pmatrix}1\\4\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}3\\0\end{pmatrix}$, $\mathbf{c}=\begin{pmatrix}0\\3\end{pmatrix}$, $\mathbf{d}=\begin{pmatrix}2\\-3\end{pmatrix}$,
$\mathbf{e}=\begin{pmatrix}-3\\-3\end{pmatrix}$, $\mathbf{f}=\begin{pmatrix}-5\\3\end{pmatrix}$, $\mathbf{g}=\begin{pmatrix}0\\-1\end{pmatrix}$

3 **a** $\begin{pmatrix}4\\1\end{pmatrix}$ **b** $\begin{pmatrix}2\\-3\end{pmatrix}$ **c** $\begin{pmatrix}-6\\-1\end{pmatrix}$ **d** $\begin{pmatrix}0\\3\end{pmatrix}$
e $\begin{pmatrix}-4\\-1\end{pmatrix}$ **f** $\begin{pmatrix}6\\-2\end{pmatrix}$ **g** $\begin{pmatrix}-4\\-4\end{pmatrix}$ **h** $\begin{pmatrix}0\\0\end{pmatrix}$

EXERCISE 26B

1 **a** $\sqrt{29}$ units **b** $\sqrt{10}$ units **c** $\sqrt{20}$ units **d** 3 units
e $\sqrt{34}$ units **f** 6 units **g** 13 units **h** $\sqrt{117}$ units
i $\sqrt{41}$ units **j** $\sqrt{45}$ units **k** $\sqrt{101}$ units **l** $\sqrt{68}$ units

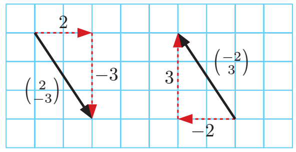
2 **a** 5 units **b** 5 units **c** 5 units **d** 5 units
Regardless of a vector's direction, if its components involve ±3 and ±4, its length is 5.

3 **a** $\begin{pmatrix}4\\-6\end{pmatrix}$ **b** $\sqrt{52}$ km ≈ 7.21 km

4 **a** $\overrightarrow{AB}=\begin{pmatrix}-3\\-2\end{pmatrix}$, $|\overrightarrow{AB}|=\sqrt{13}$ units
b $\overrightarrow{BC}=\begin{pmatrix}4\\-2\end{pmatrix}$, $|\overrightarrow{BC}|=\sqrt{20}$ units

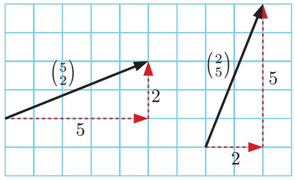
5 **a** They are parallel (in the same direction) and equal in length.
b **i** $\overrightarrow{AB}=\begin{pmatrix}4\\-1\end{pmatrix}$ **ii** $\overrightarrow{CD}=\begin{pmatrix}4\\-1\end{pmatrix}$
c $|\overrightarrow{AB}|=|\overrightarrow{CD}|=\sqrt{17}$ units

6 **a** $\overrightarrow{PQ}=\overrightarrow{SR}=\begin{pmatrix}2\\2\end{pmatrix}$, $|\overrightarrow{PQ}|=|\overrightarrow{SR}|=\sqrt{8}$ units
b $\overrightarrow{RQ}=\overrightarrow{SP}=\begin{pmatrix}-3\\2\end{pmatrix}$, $|\overrightarrow{RQ}|=|\overrightarrow{SP}|=\sqrt{13}$ units
c PQRS is a parallelogram

7 **a**


b Corresponding components have the same size, but opposite signs. The vectors have equal length, but opposite direction.
c true

EXERCISE 26C

1


The x-components are not equal, and the y-components are not equal, so the vectors are not equal.

2 $a=4$ and $b=-3$ **3** $a=0$ *or* 1 *and* $b=0$

4 $\overrightarrow{DC}=\mathbf{a}$ {as DC and AB are opposite sides of a parallelogram and so these sides are parallel and equal in length.}
Likewise $\overrightarrow{BC}=\mathbf{b}$.

5 **a** **i** **a** **ii** **b** **iii** **b** **iv** **c**
b $\mathbf{a}\neq\mathbf{b}$ as the vectors are not parallel.
c Yes, as $\triangle$ABD is equilateral.

EXERCISE 26D

1 **a** $\begin{pmatrix}6\\10\end{pmatrix}$ **b** $\begin{pmatrix}11\\11\end{pmatrix}$ **c** $\begin{pmatrix}8\\2\end{pmatrix}$ **d** $\begin{pmatrix}4\\-3\end{pmatrix}$
e $\begin{pmatrix}-3\\-3\end{pmatrix}$ **f** $\begin{pmatrix}-8\\-2\end{pmatrix}$ **g** $\begin{pmatrix}6\\7\end{pmatrix}$ **h** $\begin{pmatrix}5\\-2\end{pmatrix}$

2 **a** $\begin{pmatrix}7\\-1\end{pmatrix}$ **b** $\begin{pmatrix}7\\-1\end{pmatrix}$ **c** $\begin{pmatrix}-2\\-2\end{pmatrix}$ **d** $\begin{pmatrix}1\\3\end{pmatrix}$
e $\begin{pmatrix}10\\4\end{pmatrix}$ **f** $\begin{pmatrix}4\\-6\end{pmatrix}$ **g** $\begin{pmatrix}-12\\3\end{pmatrix}$ **h** $\begin{pmatrix}3\\0\end{pmatrix}$

3 **a** $\mathbf{a}=\mathbf{b}+\mathbf{c}$ **b** $\mathbf{q}=\mathbf{p}+\mathbf{r}$ **c** $\mathbf{p}=\mathbf{m}+\mathbf{n}$
d $\mathbf{b}=\mathbf{a}+\mathbf{d}+\mathbf{c}$ **e** $\mathbf{p}=\mathbf{q}+\mathbf{r}+\mathbf{s}$ **f** $\mathbf{a}=\mathbf{b}+\mathbf{d}+\mathbf{c}$

4 **a** $\mathbf{a}+\mathbf{b}$ **b** $\mathbf{a}+\mathbf{c}$ **c** $\mathbf{c}+\mathbf{d}$ **d** $\mathbf{a}+\mathbf{c}+\mathbf{d}$

5 **a** **c** **b** **f** **c** **g** **d** **g** **e** **g**

6 **a** $\overrightarrow{AE}$ **b** $\overrightarrow{BE}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{AE}$

7 **a** $\overrightarrow{AB}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{LD}$ **d** $\overrightarrow{SN}$ **e** $\overrightarrow{ER}$ **f** $\overrightarrow{CP}$

8 **a** $\begin{pmatrix}160\\240\end{pmatrix}$ **b** $\begin{pmatrix}-80\\120\end{pmatrix}$ **c** $\begin{pmatrix}80\\40\end{pmatrix}$
d **i** $\begin{pmatrix}160\\400\end{pmatrix}$ **ii** $\begin{pmatrix}160\\400\end{pmatrix}$ **e** ≈ 431 m
$\overrightarrow{TA}+\overrightarrow{AB}+\overrightarrow{BH}=\overrightarrow{TH}$

9 **a** $\overrightarrow{SA}=\begin{pmatrix}-2\\2\end{pmatrix}$, $\overrightarrow{AB}=\begin{pmatrix}2\\5\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}5\\3\end{pmatrix}$,
$\overrightarrow{CD}=\begin{pmatrix}4\\-2\end{pmatrix}$, $\overrightarrow{DE}=\begin{pmatrix}0\\-5\end{pmatrix}$, $\overrightarrow{EF}=\begin{pmatrix}-4\\0\end{pmatrix}$,
$\overrightarrow{FG}=\begin{pmatrix}-4\\2\end{pmatrix}$, $\overrightarrow{GS}=\begin{pmatrix}-1\\-5\end{pmatrix}$

b $\begin{pmatrix}0\\0\end{pmatrix}$

c The finishing point is the same as the starting point, so we are back where we started.

EXERCISE 26E

1 **a** $\begin{pmatrix}6\\9\end{pmatrix}$ **b** $3\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}3\times2\\3\times3\end{pmatrix}=\begin{pmatrix}6\\9\end{pmatrix}$

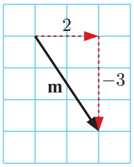
2 **a** $\begin{pmatrix}2\\4\end{pmatrix}$ **b** $\begin{pmatrix}4\\8\end{pmatrix}$ **c** $\begin{pmatrix}-1\\-2\end{pmatrix}$ **d** $\begin{pmatrix}-6\\9\end{pmatrix}$

e $\begin{pmatrix}4\\-6\end{pmatrix}$ **f** $\begin{pmatrix}2\\-1\end{pmatrix}$ **g** $\begin{pmatrix}-\frac{2}{3}\\-1\end{pmatrix}$ **h** $\begin{pmatrix}-6\\3\end{pmatrix}$

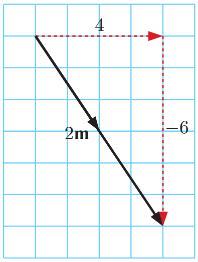
i $\begin{pmatrix}12\\18\end{pmatrix}$ **j** $\begin{pmatrix}1\\\frac{3}{2}\end{pmatrix}$ **k** $\begin{pmatrix}1\\9\end{pmatrix}$ **l** $\begin{pmatrix}-6\\3\end{pmatrix}$

m $\begin{pmatrix}8\\6\end{pmatrix}$ **n** $\begin{pmatrix}8\\0\end{pmatrix}$ **o** $\begin{pmatrix}-18\\-3\end{pmatrix}$

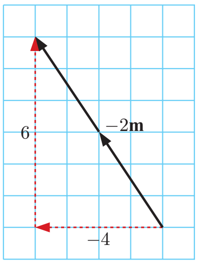
3 **a**



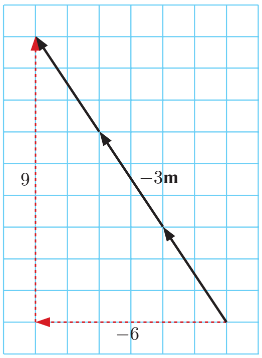
b



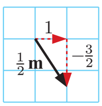
c



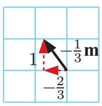
d



e



f



4 **a** $\begin{pmatrix}9\\-15\end{pmatrix}$ **b** $\begin{pmatrix}-3\\5\end{pmatrix}$ **c** $\begin{pmatrix}3\\-5\end{pmatrix}$ **d** $\begin{pmatrix}3\\-5\end{pmatrix}$

e $\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$ **f** $\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$

5 **a** $\mathbf{a}+\mathbf{0}=\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}x+0\\y+0\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{a}$

$\mathbf{0}+\mathbf{a}=\begin{pmatrix}0\\0\end{pmatrix}+\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0+x\\0+y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{a}$

b $\mathbf{a}+(-\mathbf{a})=\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-x\\-y\end{pmatrix}=\begin{pmatrix}x-x\\y-y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$

$-\mathbf{a}+\mathbf{a}=\begin{pmatrix}-x\\-y\end{pmatrix}+\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-x+x\\-y+y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}=\mathbf{0}$

6 $\overrightarrow{BA}=-\mathbf{p}$

7 **a** 2**a** **b** 2**b** **c** −**a** **d** −**b** **e** −2**a** **f** −2**b**

EXERCISE 26F

1 **a** $\begin{pmatrix}-1\\1\end{pmatrix}$ **b** $\begin{pmatrix}2\\4\end{pmatrix}$ **c** $\begin{pmatrix}9\\-1\end{pmatrix}$ **d** $\begin{pmatrix}8\\-2\end{pmatrix}$

e $\begin{pmatrix}3\\3\end{pmatrix}$ **f** $\begin{pmatrix}9\\0\end{pmatrix}$ **g** $\begin{pmatrix}-3\\2\end{pmatrix}$ **h** $\begin{pmatrix}7\\-6\end{pmatrix}$

i $\begin{pmatrix}-7\\-2\end{pmatrix}$

2 **a** $\begin{pmatrix}-2\\4\end{pmatrix}$ **b** $\begin{pmatrix}9\\-1\end{pmatrix}$ **c** $\begin{pmatrix}-7\\-3\end{pmatrix}$ **d** $\begin{pmatrix}9\\6\end{pmatrix}$

e $\begin{pmatrix}11\\2\end{pmatrix}$ **f** $\begin{pmatrix}-8\\-4\end{pmatrix}$ **g** $\begin{pmatrix}0\\\frac{7}{2}\end{pmatrix}$ **h** $\begin{pmatrix}23\\5\end{pmatrix}$

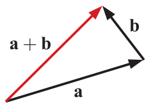
3 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{QR}$ **c** $\overrightarrow{AD}$ **d** $\overrightarrow{PV}$

4 **a** $\begin{pmatrix}6\\1\end{pmatrix}$ **b** $\begin{pmatrix}-2\\-4\end{pmatrix}$ **c** $\begin{pmatrix}8\\5\end{pmatrix}$ **d** $\begin{pmatrix}-4\\3\end{pmatrix}$

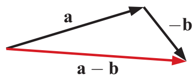
5 **a** $\overrightarrow{PR}=\mathbf{a}+\mathbf{b}$ **b** $\overrightarrow{QS}=\mathbf{b}-\mathbf{c}$

c $\overrightarrow{PS}=\mathbf{a}+\mathbf{b}-\mathbf{c}$ **d** $\overrightarrow{PM}=\frac{1}{2}(\mathbf{a}+\mathbf{b}-\mathbf{c})$

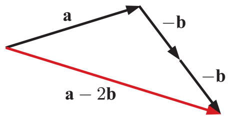
6 **a**



b



c

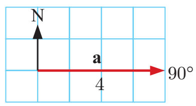


7 **a** **a** **b** **a** **c** **b** **d** **a** − **b** **e** −**b** + **a** **f** 2**a**

8 **a** **d** **b** **g** **c** **d** **d** **a** **e** −**c** **f** −**g**

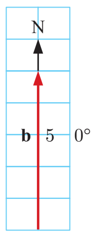
EXERCISE 26G

1 **a**



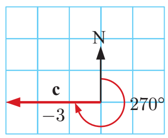
a has bearing 090°.

b



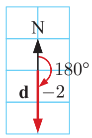
b has bearing 0°.

c



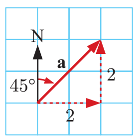
c has bearing 270°.

d



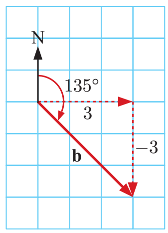
d has bearing 180°.

2 **a**



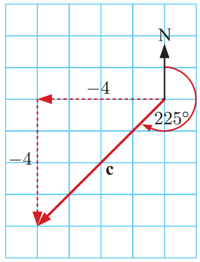
a has bearing 045°.

b



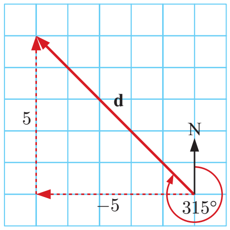
b has bearing 135°.

c



c has bearing 225°.

d

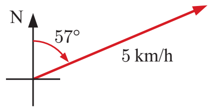


d has bearing 315°.

3 **a** 076.0° **b** 146.3° **c** 306.9° **d** 206.6°

4 **a** $|\mathbf{a}| = \sqrt{10}$ units, bearing is 341.6°
b $|\mathbf{b}| = \sqrt{13}$ units, bearing is 158.2°
c $|\mathbf{c}| = \sqrt{10}$ units, bearing is 251.6°
d $|\mathbf{d}| = \sqrt{13}$ units, bearing is 123.7°

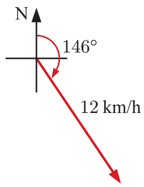
5 **a**



Scale: 1 cm ≡ 2 km/h

b velocity vector is $\begin{pmatrix} 4.19 \\ 2.72 \end{pmatrix}$

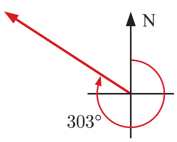
6 **a**



Scale: 1 cm ≡ 6 km/h

b $\mathbf{v} = \begin{pmatrix} 6.71 \\ -9.95 \end{pmatrix}$

7 **a**



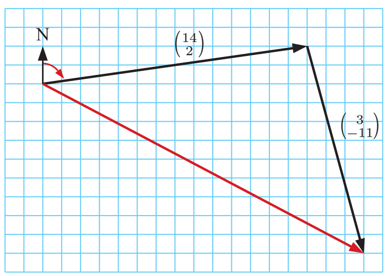
Scale: 1 cm ≡ 4 km

b $\mathbf{v} = \begin{pmatrix} -6.71 \\ 4.36 \end{pmatrix}$

EXERCISE 26H

1 **a** $\begin{pmatrix} 0 \\ 10 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ 10 \end{pmatrix}$ **c** $\sqrt{125}$ km
d ≈ 333.4°

2 **a**

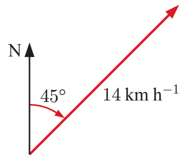


b $\begin{pmatrix} 17 \\ -9 \end{pmatrix}$ **c** $\sqrt{370} \approx 19.2$ km **d** ≈ 117.9°

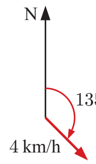
3 **a** $\sqrt{5} \approx 2.24$ m/s **b** $\theta \approx 26.6°$ **c** ≈ 116.6°

4 **a** 15 km/h
b Yes, if she walks from the back forwards to the front, her relative speed is greater than the ferry's speed.

5 **a** **i**



ii



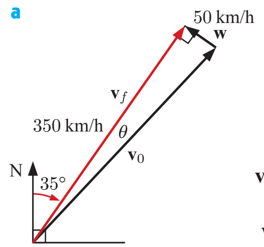
Scale: 1 cm ≡ 5 km/h

b **i** **Hint:** Yacht's velocity vector is $\begin{pmatrix} 14\cos 45° \\ 14\sin 45° \end{pmatrix}$

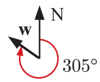
Wake's velocity vector is $\begin{pmatrix} 4\cos 45° \\ -4\sin 45° \end{pmatrix}$

c ≈ 14.6 km/h with bearing 060.9°

6 **a**



Scale: 1 cm ≡ 100 km/h

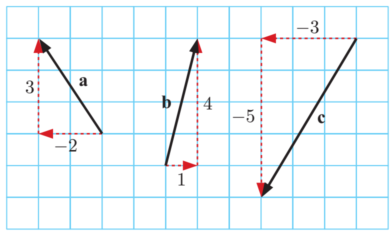


$\mathbf{v}_0$ = plane's original vector (no wind)
$\mathbf{w}$ = wind vector
$\mathbf{v}_f$ = plane's actual velocity vector

b $\mathbf{v}_0 \approx \begin{pmatrix} 242 \\ 258 \end{pmatrix}$

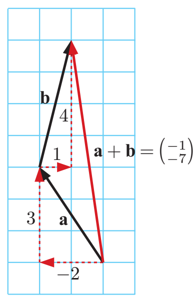
REVIEW SET 26A

1



2 **a** $\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **b** $\mathbf{e} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

3 **a**

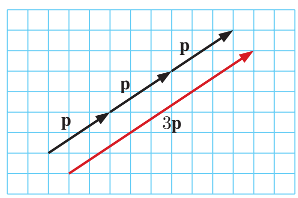


b **i** $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ **ii** $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ **iii** $\begin{pmatrix} 2 \\ 12 \end{pmatrix}$

4 **a** $\sqrt{53}$ units **b** 5 units **c** $\sqrt{73}$ units

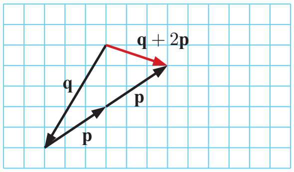
5 $a = 3, \ b = 7$

6 **a**



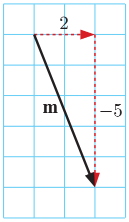
b **i** $\begin{pmatrix} -6 \\ -10 \end{pmatrix}$ **ii** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ **iii** $\begin{pmatrix} 9 \\ 12 \end{pmatrix}$

c



$\mathbf{q} + 2\mathbf{p} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

7 **a**



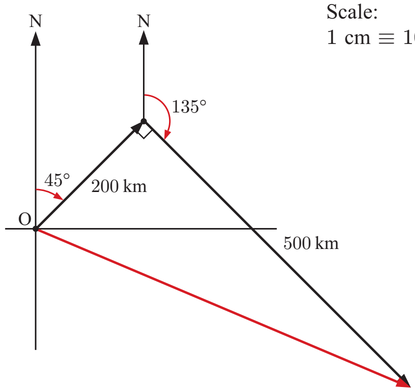
b $\sqrt{29}$ units
c $\approx 158.2°$

8 **a** **i** $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ **ii** $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$ **b** $\overrightarrow{AC}$ **c** $\sqrt{34}$ units

9 **a** $\approx 018.4°$ **b** $\approx 201.8°$

10 **a**

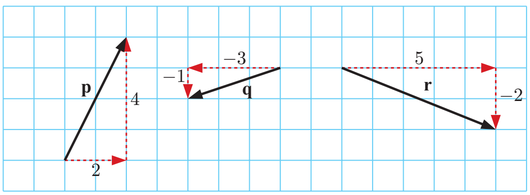
Scale:
1 cm ≡ 100 km



b ≈ 539 km **c** $\approx 293.2°$

REVIEW SET 26B

1



2 **a** $\mathbf{m} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$ **b** $\mathbf{n} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$

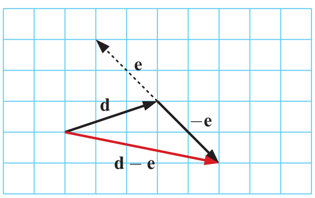
3 **a** $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

b $|\overrightarrow{PQ}| = \sqrt{13}$ units, $|\overrightarrow{QR}| = \sqrt{34}$ units

4 **a** $\begin{pmatrix} 11 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 10 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$

5 **a** $\begin{pmatrix} 18 \\ -15 \end{pmatrix}$ **b** $\begin{pmatrix} -12 \\ 10 \end{pmatrix}$ **c** $\begin{pmatrix} -6 \\ 5 \end{pmatrix}$ **d** $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6 **a**



b $\mathbf{d} - \mathbf{e} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ **c** **i** $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ **ii** $\begin{pmatrix} 18 \\ -2 \end{pmatrix}$

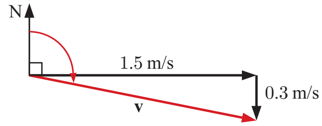
7 **a** $-\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $\frac{1}{2}\mathbf{q} - \mathbf{p}$ **d** $\mathbf{q}$

8 **a** $\overrightarrow{TR} = \mathbf{e} + \mathbf{d}$ **b** $\overrightarrow{PR} = \mathbf{c} + \mathbf{d}$ **c** $\overrightarrow{PT} = \mathbf{c} - \mathbf{e}$
d $\overrightarrow{TQ} = \mathbf{e} + \mathbf{d} + \mathbf{b}$

9 $\begin{pmatrix} -3.54 \\ 3.54 \end{pmatrix}$

10 **a**

Scale:
1 cm ≡ 0.5 m/s



b speed ≈ 1.53 m/s with bearing $101.3°$
c 567 sec (≈ 9.44 min) **d** 170 m to the south

11 **Hint:** Let $\overrightarrow{BM} = \mathbf{a}$ and $\overrightarrow{BN} = \mathbf{b}$. Find $\overrightarrow{MN}$ and $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.